Integration by parts on the law of the reflecting Brownian motion

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Abstract

We prove an integration by parts formula on the law of the reflecting Brownian motion $X := |B|$ in the positive half line, where B is a standard Brownian motion. In other terms, we consider a perturbation of X of the form $X^\epsilon = X + \epsilon h$ with h smooth deterministic function and $\epsilon > 0$ and we differentiate the law of X^ϵ at $\epsilon = 0$. This infinitesimal perturbation changes drastically the set of zeros of X for any $\epsilon > 0$. As a consequence, the formula we obtain contains an infinite dimensional generalized functional in the sense of Schwartz, defined in terms of Hida's renormalization of the squared derivative of B and in terms of the local time of X at 0. We also compute the divergence on the Wiener space of a class of vector fields not taking values in the Cameron-Martin space.

1 Introduction

In this paper we want to prove an infinite dimensional integration by parts formula with respect to the law of the reflecting Brownian motion (RBM) $X_\theta := |B_\theta - a|$, $\theta \in [0,1]$, where B is a standard Brownian motion and $a \in \mathbb{R}$.

Integration by parts formulae on infinite dimensional probability measures are an important tool in a number of topics in Stochastic Analysis.

Typically, given a stochastic process X, one considers the law of an infinitesimal variation $X^\varepsilon := X + \varepsilon h$, where h is a process in a suitable class, and one tries to differentiate the law of X^ε w.r.t. ε at $\varepsilon = 0$. In most cases one exploits a quasi-invariance property, i.e. one chooses h in such a way that the law of X^ε is absolutely continuous w.r.t. the law of X: see the monograph [10]. If this is possible, then the problem is reduced to differentiate the density.

This project has been implemented e.g. for a large class of diffusions in $\mathbb{R}^d$ or in Riemannian manifolds, see e.g. [4], [7] and [6], and for Poisson measures, see e.g. [1]. Recently integration by parts for a class of processes with values in $(0, \infty)$, the Bessel bridges of dimension $d \geq 3$, have been computed: see [12] and [13].

However, the case of processes with a non-trivial behavior at a boundary remains an open problem. A typical example of such processes is the RBM, which takes values in $[0, \infty)$ and has a local time at the boundary $\{0\}$.

In §4 of [2] J.-M. Bismut developed a stochastic calculus of variations for the RBM $X = |B - a|$, with the aim of studying transition probabilities of boundary processes associated with diffusions. However the results of [2] concern only variations $X + \varepsilon h$ of X with the crucial property $\{t : h_t = 0\} = \{t : X_t = 0\}$. In this case the quasi-invariance property holds. Notice that h is necessarily a non-deterministic process.

In this paper we consider perturbations $X^\varepsilon = X + \varepsilon h$ of $X = |B - a|$, with h smooth deterministic function with compact support in $(0, 1)$. In this case, the approach based on the quasi-invariance fails, since the law of X^ε is not absolutely continuous w.r.t. the law of X if $\varepsilon > 0$ and h not identically 0: see the argument at the end of this introduction.

As a consequence of the lack of quasi-invariance, the integration by parts formula we obtain does not contain only the law of X times suitable densities, as it is usual in the Malliavin calculus, see e.g. [7], but also an infinite dimensional generalized functional, in the sense of Schwartz: see Theorem 2.3 below.

This generalized functional is defined in terms of Hida's square of the white noise, i.e. a renormalization of the squared derivative of B, defined e.g. in [5], and in terms of the local time of B at 0: see Theorem 2.1 below.

It turns out that this problem is closely related with the computation of the divergence on the Wiener space of a class of vector fields not taking values in the Cameron-Martin space. The divergence of vector fields taking values in the Cameron-Martin space is typically an L^p-variable: see the monograph [7].

The divergence we obtain is not an L^p-variable but a generalized functional related with the one discussed above: see Theorem 2.2 below.

We show now that the law of X^ε is not absolutely continuous w.r.t. the law of $X = |B-a|$ if $\varepsilon > 0$ and h is not identically 0. In the case $\min h < 0$, with positive probability $\min X^\varepsilon < 0$, while $X \geq 0$ almost surely, so we can suppose $h \geq 0$. Let I be a non-empty interval where $h > 0$ and define the set of continuous paths over $[0,1]$:

$$\Omega^\varepsilon := \left\{ \omega : \min_{\tau \in I} (\omega_\tau - \varepsilon h_\tau) = 0 \right\}.$$

We claim that $\mathbb{P}(X^\varepsilon \in \Omega^\varepsilon) > 0$ while $\mathbb{P}(X \in \Omega^\varepsilon) = 0$.

Indeed, $X^\varepsilon \in \Omega^\varepsilon$ if and only if there exists $\tau \in I$ such that $B_\tau = a$. Since this event has positive probability, then $\mathbb{P}(X^\varepsilon \in \Omega^\varepsilon) > 0$. On the other hand:

$$\mathbb{P}(X \in \Omega^\varepsilon) = \mathbb{P}(B - a \in \Omega^\varepsilon) + \mathbb{P}(a - B \in \Omega^\varepsilon).$$

By the Girsanov Theorem, the law of $(B_\tau - a - \varepsilon h_\tau : \tau \in I)$ is absolutely continuous w.r.t. the law of $(B_\tau : \tau \in I)$, with Radon-Nikodym density ρ. In particular:

$$\mathbb{P}(B - a \in \Omega^\varepsilon) = \mathbb{E}\left[\rho \, 1_{(\min_I B = 0)}\right],$$

but the r.v. $\min_I B$ has a continuous density, so that $\mathbb{P}(\min_I B = 0) = 0 = \mathbb{P}(B - a \in \Omega^\varepsilon)$. Arguing analogously for $\mathbb{P}(a - B \in \Omega^\varepsilon)$ we obtain that $\mathbb{P}(X \in \Omega^\varepsilon) = 0$.

2 Main results

Let $(B_\theta : \theta \in [0,1])$ be a standard Brownian motion and $C := \{k : [0,1] \mapsto \mathbb{R}$ continuous, $k_0 = 0\}$. We denote by μ the law of B on C: then (C, μ) is the classical Wiener space. We introduce $L := L^2(0,1)$ with scalar product:

$$\langle h, k \rangle := \int_0^1 k_\theta \, h_\theta \, d\theta, \qquad \|h\|^2 := \langle h, h \rangle, \qquad h, k \in L.$$

We consider the following function space on L: the set $\mathrm{Lip}_e(L)$ of $F : L \mapsto \mathbb{R}$ such that:

$$\exists\, c > 0 : \quad |F(h) - F(k)| \leq e^{c\|h\|} \, \|h - k\|, \qquad h, k \in L.$$

Notice that all functions in $\mathrm{Lip}_e(L)$ are Lipschitz on balls of L, with constant growing at most exponentially with the radius.

Let $(\rho_\epsilon)_{\epsilon>0}$ be a family of smooth symmetric mollifiers on $\mathbb{R}$, i.e.

$$\rho_\epsilon := \frac{1}{\epsilon}\rho\left(\frac{\cdot}{\epsilon}\right), \quad \rho \in C_c^\infty(-1,1), \quad \rho \geq 0, \quad \int_0^1 \rho\, dx = 1, \quad \rho(x) = \rho(-x).$$

We denote for $\theta \in [0,1]$, $\ell \in C$:

$$\ell_{\epsilon,\theta} = (\rho_\epsilon * \ell)_\theta = \int_0^1 \rho_\epsilon(\sigma-\theta)\,\ell_\sigma\, d\sigma,$$

$$\dot{\ell}_{\epsilon,\theta} = \ell'_{\epsilon,\theta} = \frac{d}{d\theta}\,\ell_{\epsilon,\theta} = (-\rho'_\epsilon * \ell)_\theta.$$

With this definition, we denote throughout the paper:

$$:\dot{B}^2_{\epsilon,\theta}: \stackrel{\mathrm{def}}{=} \left(\dot{B}_{\epsilon,\theta}\right)^2 - \mathbb{E}\left[\left(\dot{B}_{\epsilon,\theta}\right)^2\right], \qquad \theta \in [0,1].$$

Here we regularize B, we differentiate the regularization $B_{\epsilon,\cdot}$, we square the derivative and finally we center this r.v. by subtracting the mean.

Let $(L^a_\theta : \theta \in [0,1])$ denote the local time of B at $a \in \mathbb{R}$, defined by the occupation times formula:

$$\int_0^\theta \psi(s,B_s)\, ds = \int_{\mathbb{R}} \int_0^\theta \psi(s,a)\, dL^a_s\, da, \qquad \theta \in [0,1], \tag{2.1}$$

for all bounded Borel $\psi : [0,\infty) \times \mathbb{R} \mapsto \mathbb{R}$, see Chapter VI of [9]. Finally, let $C_c(0,1)$ denote the space of continuous h with compact support in $(0,1)$ and $C^2_c(0,1)$ the set of $h \in C_c(0,1)$ with continuous second derivative.

Then we can state the first Theorem:

Theorem 2.1. *For all $h \in C_c(0,1)$ and $F \in \mathrm{Lip}_e(L)$, there exists the limit:*

$$\lim_{\epsilon\to 0} \mathbb{E}\left[F(B)\int_0^1 h_\theta\ :\dot{B}^2_{\epsilon,\theta}:\ dL^a_\theta\right] \tag{2.2}$$

$$=: \mathbb{E}\left[F(B)\int_0^1 h_\theta\ :\dot{B}^2_\theta:\ dL^a_\theta\right].$$

In the r.h.s. of (2.2), $:\dot{B}^2_\theta:$ is the renormalization of the square of the derivative of B, i.e. Hida's square of the white noise: since B is not differentiable, the expression $\dot{B}^2$ is not well defined; nevertheless, subtracting to

$\dot{B}^2$ a diverging constant, we obtain convergence to a generalized functional on the Wiener space. This is made rigorous by the White Noise Analysis, a generalization to infinite dimension of Schwartz's Theory of Distributions, see e.g. [5]. However the convergence of the particular functional defined by (2.2) does not seem to be covered by the existing theorems in the literature, because of the integration w.r.t. the local time process.

Notice that Theorem 2.1 defines the r.h.s. of (2.2) through the limit in the l.h.s.: this can be unsatisfactory and it seems reasonable to look for a direct way of computing the functional on $F \in \mathrm{Lip}_e(L)$: this is done in the last result of the paper, Corollary 6.1 below. We remark that it is crucial for the application to the RBM given in Theorem 2.3 below that the limit in (2.2) exists for a large class of Lipschitz-continuous functions on L, like $\mathrm{Lip}_e(L)$.

Before stating the second Theorem, we need a few more notations. We introduce the Cameron-Martin space $H^1 := \{h \in C : h' \in L,\ h(0) = 0\}$. We also consider a second function space on L: the set $C^1_e(L)$ of all $F \in \mathrm{Lip}_e(L)$ with continuous Fréchet differential $\nabla F : L \mapsto L$. Notice that ∇F satisfies:

$$\exists\, c > 0 : \quad \|\nabla F(h)\| \,\le\, e^{c\|h\|}, \qquad h \in L.$$

For any $\varphi : \mathbb{R} \mapsto \mathbb{R}$ with continuous continuous derivative and any smooth deterministic $h : (0,1) \mapsto \mathbb{R}$ with compact support, we can define the following vector field over C:

$$\mathcal{K} : C \mapsto C, \qquad \mathcal{K}(\omega) \,:=\, h\, \varphi'(\omega).$$

Notice that $\mathcal{K}$ does not take values in the Cameron-Martin space H^1, since in general the regularity of $\varphi'(\omega)$ is not better than that of $\omega \in C$. Therefore the divergence of $\mathcal{K}$ on the Wiener space can not be computed with the classical theory of the Malliavin calculus, see [7]. One of the results of this paper, given in Theorem 2.2, is the computation of this non-classical divergence.

During the paper we shall consider φ in the class:

$$\begin{aligned}\mathrm{Conv}(\mathbb{R}) \;:=\; \Big\{ & \varphi_1 - \varphi_2, \qquad \varphi_i : \mathbb{R} \mapsto \mathbb{R} \quad \text{convex}, \\ & \exists\, c > 0 \;:\; \quad |\varphi_i'(x)| \,\le\, e^{c|x|}, \quad \forall x \in \mathbb{R},\ i = 1,2 \Big\}.\end{aligned}$$

If $F \in C^1_e(L)$, $h \in C_c(0,1)$ and $\varphi \in \mathrm{Conv}(\mathbb{R})$, then we can define the directional derivative of F at $\omega \in C$ along $\mathcal{K}(\omega)$:

$$\partial_{h\,\varphi'(\omega)} F(\omega) \,:=\, \lim_{\epsilon \to 0} \frac{F(\omega + \epsilon\, h\, \varphi'(\omega)) - F(\omega)}{\epsilon}.$$

Theorem 2.2. *For all $\varphi \in \mathrm{Conv}(\mathbb{R})$, $h \in C^2_c(0,1)$ and $F \in C^1_e(L)$ the following integration by parts formula holds:*

$$\begin{aligned}\mathbb{E}\left[\partial_{h\,\varphi'(B)} F(B)\right] \;&=\; -\,\mathbb{E}\left[F(B)\int_0^1 h''_\theta\,\varphi(B_\theta)\,d\theta\right] \\ &\quad + \int_{\mathbb{R}} \mathbb{E}\left[F(B)\int_0^1 h_\theta \;:\dot{B}^2_\theta:\; dL^a_\theta\right]\varphi''(da).\end{aligned} \tag{2.3}$$

We notice that an infinitesimal transformation along $\mathcal{K}$ does not preserve the absolute-continuity class of the Wiener measure. For instance, in the case $\varphi(r) = r^2$, the infinitesimal transformation along $\mathcal{K}$ is $B \mapsto B + \epsilon h\varphi'(B) = B(1+2\epsilon h)$ and it is well known that the laws of B and $B(1+2\epsilon h)$ are singular if $\epsilon h \neq 0$. This explains why the r.h.s. of (2.3) contains a term, the second one, which is not a measure but a generalized functional over C. We treat the case $\varphi(r) = r^2$ and $h \equiv 1$ separately in section 7.

We can now turn to the reflecting Brownian motion $X := |B - a|$, for some $a \geq 0$. For all smooth $f : C \mapsto \mathbb{R}$ and $h \in C^2_c(0,1)$, by applying (2.3) to $F(\omega) := f(|\omega - a|)$ we obtain the following:

Theorem 2.3. *We set $X := |B - a|$ and we denote by $(\ell^0_\theta : \theta \in [0,1])$ the local time of X at 0. Then for all $h \in C^2_c(\mathbb{R})$ and $f \in C^1_e(L)$:*

$$\mathbb{E}\left[\partial_h f(X)\right] \;=\; -\,\mathbb{E}\left[f(X)\int_0^1 h''_\theta\,X_\theta\,d\theta\right] + \mathbb{E}\left[f(X)\int_0^1 h_\theta \;:\dot{B}^2_\theta:\; d\ell^0_\theta\right]. \tag{2.4}$$

By Tanaka's formula $\ell^0 \equiv 2\,L^a$, see Chapter VI of [9]. Moreover $f(X) = f(|B - a|)$. Therefore the second term in the r.h.s. of (2.4) is defined by (2.2).

We give a heuristic argument motivating the result of Theorem 2.2. If $F \in C^1_e(L)$, then the classical integration by parts formula for the Wiener measure states:

$$\mathbb{E}\left[\partial_h F(B)\right] \;=\; \mathbb{E}\left[F(B)\int_0^1 h'_\theta\,dB_\theta\right],$$

for all deterministic $h \in H^1$, i.e. such that $h' \in L^2(0,1)$ and $h(0) = 0$.

Consider now a process $(\mathcal{K}_\theta(B) : \theta \in [0,1])$ such that:

1. $\mathcal{K}_\theta = \int_0^\theta \dot{\mathcal{K}}_s\,ds$, with $\dot{\mathcal{K}}(B)$ adapted and uniformly bounded.

2. there exists a continuous $(\mathcal{Q}_{\theta,\theta'}(\omega) : \theta, \theta' \in [0,1])$ s.t. for all $k \in H^1$:

$$\frac{d}{d\varepsilon} \mathcal{K}_\theta(\omega + \varepsilon k)\bigg|_{\varepsilon=0} = \int_0^1 \mathcal{Q}_{\theta,\theta'}(\omega)\, k_{\theta'}\, d\theta', \qquad \theta \in [0,1],\ \omega \in L.$$

Then the integration by parts formula becomes:

$$\mathbb{E}\left[\partial_{\mathcal{K}(B)} F(B)\right] = \mathbb{E}\left[\left(\int_0^1 \dot{\mathcal{K}}_\theta(B)\, dB_\theta - \int_0^1 \mathcal{Q}_{\theta,\theta}(B)\, d\theta\right) F(B)\right].$$

We set now $\mathcal{K}_\theta(\omega) := h_\theta\, \varphi'(\omega_\theta)$, where $h \in C_c^2(0,1)$ and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is twice continuously differentiable with bounded derivatives. In this case $\mathcal{K}_\cdot$ is adapted but not a.s. in H^1, since $\varphi'(B_\cdot)$ has a non-trivial martingale part. Moreover for all $k \in H^1$:

$$\frac{d}{d\varepsilon} \mathcal{K}_\theta(\omega + \varepsilon k)\bigg|_{\varepsilon=0} = h_\theta\, \varphi''(\omega_\theta)\, k_\theta, \qquad \theta \in [0,1],$$

so that $\mathcal{Q}_{\theta,\theta'} = h_\theta\, \varphi''(\omega_\theta)\, \delta(\theta - \theta')$, where δ is the Dirac function. In particular $\mathcal{Q}_{\theta,\theta} = h_\theta\, \varphi''(\omega_\theta)\, \delta(0)$ is ill-defined, since $\delta(0) = \infty$. However, arguing formally, we can write:

$$\int_0^1 \mathcal{Q}_{\theta,\theta}(B)\, d\theta = \int_0^1 h_\theta\, \varphi''(B_\theta)\, \delta(0)\, d\theta.$$

Moreover, pretending that $B_\cdot$ is differentiable and $dB_\theta = \dot{B}_\theta\, d\theta$, we obtain:

$$\begin{aligned} \int_0^1 \dot{\mathcal{K}}_\theta(B)\, dB_\theta &= \int_0^1 \frac{d}{d\theta}\left[h\, \varphi'(B)\right] \dot{B}\, d\theta \\ &= \int_0^1 h'\, \varphi'(B)\, \dot{B}\, d\theta + \int_0^1 h\, \varphi''(B)\, \dot{B}^2\, d\theta. \end{aligned}$$

Since $\varphi'(B_\theta)\, \dot{B}_\theta = \frac{d}{d\theta}\varphi(B_\theta)$, integrating by parts over $[0,1]$ in the first term of this sum, we obtain:

$$\begin{aligned} &\int_0^1 \dot{\mathcal{K}}_\theta(B)\, dB_\theta - \int_0^1 \mathcal{Q}_{\theta,\theta}(B)\, d\theta \\ &= -\int_0^1 h''\, \varphi(B)\, d\theta + \int_0^1 h\ :\dot{B}^2:\ \varphi''(B)\, d\theta, \end{aligned}$$

where $:\dot{B}^2: = \dot{B}^2 - \delta(0)$. In order to get (2.3) we apply the occupation times formula (2.1) formally:

$$\int_0^1 h \ :\dot{B}^2: \ \varphi''(B)\, d\theta \ = \ \int_{\mathbb{R}} \left[\int_0^1 h \ :\dot{B}^2: \ dL^a_\theta \right] \varphi''(da).$$

The paper is organized as follows. In section 3 we prove that Theorems 2.1 and 2.2 holds for all F in a suitable space of test functions. In section 4 we introduce an infinite dimensional Sobolev space on C and several related functional analytical tools. We prove Theorems 2.1, 2.2 and 2.3 in section 5, postponing the proof of the main estimate, given in Lemma 5.3, to section 6. Finally, in section 7 we discuss the particular case of quadratic φ.

We denote by $C_b(\mathbb{R})$ the space of bounded continuous real functions on $\mathbb{R}$ and by $C^k_b(\mathbb{R})$ the set of $f \in C_b(\mathbb{R})$ such the i-th derivative of f belongs to $C_b(\mathbb{R})$ for all $i = 1, \ldots, k$.

We will use the letter κ to denote positive finite constants whose exact value may change from line to line.

3 White noise calculus

In this section we prove that formulae (2.2) and (2.3) hold for all F in the following space of test functions over C:

$$\mathrm{Exp}(C) \ := \ \mathrm{Span}\{\exp(\langle \cdot, k\rangle) : \ k \in C\},$$

i.e. we prove the following:

Proposition 3.1. *Let $h \in C_c(0,1)$, and $a \in \mathbb{R}$. Then for all $F \in \mathrm{Exp}(C)$ the limit in (2.2) exists.*

Proposition 3.2. *Let $h \in C^2_c(0,1)$ and $\varphi \in \mathrm{Conv}(\mathbb{R})$. Then for all $F \in \mathrm{Exp}(C)$ formula (2.3) holds.*

Propositions 3.1 and 3.2 show that Theorems 2.1 and 2.2 hold for all F in a suitable space of test functions. The proof of this result is elementary and based only on the Cameron-Martin theorem and on Itô's formula.

We introduce the operator:

$$Q : L \mapsto L, \qquad Qk_\theta \ := \int_0^1 \theta \wedge \sigma \, k_\sigma \, d\sigma, \qquad \theta \in [0,1].$$

The law of B in L is the Gaussian measure with mean 0 and covariance operator Q, i.e.

$$\mathbb{E}\left[e^{\langle B,k\rangle}\right] \,=\, e^{\frac{1}{2}\langle Qk,k\rangle}, \qquad k \in L.$$

By the uniqueness of the Laplace transform, we obtain the following version of the Cameron-Martin formula: for all bounded Borel $\Phi : C \mapsto \mathbb{R}$

$$\mathbb{E}\left[\Phi(B)\, e^{\langle B,k\rangle}\right] \,=\, e^{\frac{1}{2}\langle Qk,k\rangle}\, \mathbb{E}[\Phi(B + Qk)], \quad k \in C. \tag{3.1}$$

This simple formula is crucial in White Noise Analysis, in particular in the definition of the so called $\mathcal{S}$-transform: see e.g. chapter 2 of [5].

We set for $\epsilon < \min\{\theta, 1-\theta\}$:

$$c_{\epsilon,\theta} \,:=\, \mathbb{E}\left[\dot{B}^2_{\epsilon,\theta}\right] \,=\, \langle Q\rho'_\epsilon(\cdot - \theta), \rho'_\epsilon(\cdot - \theta)\rangle. \tag{3.2}$$

We also define:

$$\lambda(\theta, x, y) \,:=\, x^2 + \frac{x\,y}{\theta} + \frac{y^2 - \theta}{4\theta^2}, \qquad \theta \in (0,1),\ x, y \in \mathbb{R}. \tag{3.3}$$

The proof of Proposition 3.1 is based on the following:

Lemma 3.3. *For all $\psi \in C_b(\mathbb{R})$, $k \in C$, $K := Qk$, $\theta \in [\epsilon, 1-\epsilon] \subset (0,1)$:*

$$\mathbb{E}\left[\psi(B_\theta)\ :\dot{B}^2_{\epsilon,\theta}:\ e^{\langle B,k\rangle}\right] \,=\, e^{\frac{1}{2}\langle Qk,k\rangle}\, \mathbb{E}\left[\psi(B_\theta + K_\theta)\ \lambda(\theta, K'_{\epsilon,\theta}, B_\theta)\right]. \tag{3.4}$$

Proof. We fix $\theta \in (0,1)$ and set

$$\ell_\sigma \,:=\, 1_{[0,\theta]}(\sigma)\,\frac{\sigma}{\theta} + 1_{(\theta,1]}(\sigma), \qquad \beta_\sigma \,:=\, B_\sigma - B_\theta\, l_\sigma, \quad \sigma \in [0,1].$$

Then β and B_θ are independent, i.e. for all $\Phi : C \mapsto \mathbb{R}$ bounded Borel:

$$\mathbb{E}[\,\psi(B_\theta)\,\Phi(B)] \,=\, \int_{\mathbb{R}} \mathcal{N}(0,\theta)(dy)\ \psi(y)\, \mathbb{E}\left[\Phi\left(\beta + y\,\ell\right)\right].$$

Then by (3.1):

$$\begin{aligned}
&\mathbb{E}\left[\,\psi(B_\theta)\ :\dot{B}^2_{\epsilon,\theta}:\ e^{\langle B,k\rangle}\right] \,= \\
&=\, e^{\frac{1}{2}\langle Qk,k\rangle}\mathbb{E}\left[\psi(B_\theta + K_\theta)\ \left[\left((B+K)'_{\epsilon,\theta}\right)^2 - c_{\epsilon,\theta}\right]\right] \\
&=\, e^{\frac{1}{2}\langle Qk,k\rangle}\int_{\mathbb{R}} \mathcal{N}(0,\theta)(dy)\ \psi(y + K_\theta)\left[\mathbb{E}\left[\left((\beta + y\,\ell + K)'_{\epsilon,\theta}\right)^2\right] - c_{\epsilon,\theta}\right].
\end{aligned}$$

Since $\theta \in [\epsilon, 1-\epsilon]$, we have:

$$\ell'_{\epsilon,\theta} \,=\, (\rho_\epsilon * \ell')_\theta \,=\, \int \rho_\epsilon(\sigma-\theta)\,\frac{1}{\theta}\,1_{[0,\theta]}(\sigma)\,d\sigma \,=\, \frac{1}{2\theta}.$$

Then easy computations yield:

$$\begin{aligned}
&\mathbb{E}\left[\left((\beta+y\,\ell+K)'_{\epsilon,\theta}\right)^2\right] - c_{\epsilon,\theta} \\
&= \left(K'_{\epsilon,\theta}\right)^2 \,+\, 2y\,K'_{\epsilon,\theta}\ell'_{\epsilon,\theta} \,+\, y^2\left(\ell'_{\epsilon,\theta}\right)^2 \,+\, \mathbb{E}\left[\left(\beta'_{\epsilon,\theta}\right)^2\right] - c_{\epsilon,\theta} \\
&= \left(K'_{\epsilon,\theta}\right)^2 \,+\, \frac{y}{\theta}\,K'_{\epsilon,\theta} \,+\, \frac{1}{4\theta^2}\,(y^2-\theta).
\end{aligned}$$

This yields the thesis. $\square$

Proof of Proposition 3.1. Let $h \in C_c(0,1)$. Multiplying (3.4) by h_θ and integrating in θ we have:

$$\begin{aligned}
&\mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta\ :\dot B^2_{\epsilon,\theta}:\ \psi(B_\theta)\,d\theta\right] \\
&= \mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta\,\lambda(\theta, K'_{\epsilon,\theta}, B_\theta-K_\theta)\,\psi(B_\theta)\,d\theta\right].
\end{aligned}$$

By the occupation times formula (2.1), this implies for all $a \in \mathbb{R}$:

$$\mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta :\dot B^2_{\epsilon,\theta}:\ dL^a_\theta\right] \,=\, \mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta\,\lambda(\theta, K'_{\epsilon,\theta}, a-K_\theta)\,dL^a_\theta\right].$$

Since for all $k \in C$ we have $K'_{\epsilon,\theta} \to K'_\theta$ as $\epsilon \to 0$, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta :\dot B^2_\theta:\ dL^a_\theta\right] \,:=\, \lim_{\epsilon\to 0}\mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta :\dot B^2_{\epsilon,\theta}:\ dL^a_\theta\right] \\
&= \mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta\,\lambda(\theta, K'_\theta, a-K_\theta)\,dL^a_\theta\right]. \quad \square \qquad (3.5)
\end{aligned}$$

In Lemma 3.3 we have in fact computed the Laplace transform of the distribution on the Wiener space defined by (2.2):

Corollary 3.4. *For all $a \in \mathbb{R}$, $h \in C_c(0,1)$ and $k \in C$:*

$$\mathbb{E}\left[e^{\langle B,k\rangle}\int_0^1 h_\theta \ :\dot{B}_\theta^2:\ dL_\theta^a\right]$$

$$= e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 h_\theta \frac{e^{-(a-K_\theta)^2/2\theta}}{\sqrt{2\pi\theta}}\,\lambda(\theta,K'_\theta,a-K_\theta)\,d\theta,$$

where λ is defined in (3.3).

We turn now to the proof of Proposition 3.2. For $\Psi_k := \exp(\langle\cdot,k\rangle)$, $k \in C$, we have:

$$\partial_{h\,\varphi'(\omega)}\Psi_k(\omega) \ = \ \lim_{\epsilon\to 0}\frac{\Psi_k(\omega+\epsilon\,h\,\varphi'(\omega))-\Psi_k(\omega)}{\epsilon} \ = \ \Psi_k(\omega)\int_0^1 k_\theta h_\theta\,\varphi'(\omega_\theta)\,d\theta.$$

Therefore, by (3.1), the l.h.s. of (2.3) with $F = \Psi_k$ is equal to:

$$\begin{aligned}\mathbb{E}\left[\partial_{h\,\varphi'(B)}\Psi_k(B)\right] \ &= \ \mathbb{E}\left[\Psi_k(B)\int_0^1 k_\theta\,h_\theta\,\varphi'(B_\theta)\,d\theta\right] \\ &= \ e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 h_\theta\,k_\theta\,\mathbb{E}\left[\varphi'(B_\theta+K_\theta)\right]\,d\theta.\end{aligned} \tag{3.6}$$

The proof of Proposition 3.2 is based on the following easy application of Itô's formula.

Lemma 3.5. *For all $\varphi \in C_b^2(\mathbb{R})$, $k \in C$, $K := Qk$ and $\theta \in (0,1)$ we have:*

$$k_\theta\,\mathbb{E}\left[\varphi'(B_\theta+K_\theta)\right] \tag{3.7}$$

$$= -\frac{d^2}{d\theta^2}\mathbb{E}\left[\varphi(B_\theta+K_\theta)\right] + \mathbb{E}\left[\varphi''(B_\theta+K_\theta)\ \lambda(\theta,K'_\theta,B_\theta)\right].$$

Proof. By approximation, it is enough to consider the case $\varphi \in C_b^4(\mathbb{R})$. By Itô's formula:

$$\begin{aligned}\varphi(B_\theta+K_\theta) \ &= \ \varphi(0) + \int_0^\theta \varphi'(B_\sigma+K_\sigma)\,(dB_\sigma \ + \ K'_\sigma\,d\sigma) \\ &\quad + \frac{1}{2}\int_0^\theta \varphi''(B_\sigma+K_\sigma)\,d\sigma.\end{aligned}$$

Taking expectation and differentiating in θ we obtain:

$$\frac{d}{d\theta}\,\mathbb{E}\left[\varphi(B_\theta+K_\theta)\right] \;=\; K'_\theta\,\mathbb{E}\left[\varphi'(B_\theta+K_\theta)\right] \,+\, \frac{1}{2}\,\mathbb{E}\left[\varphi''(B_\theta+K_\theta)\right].$$

By iteration of this formula we obtain:

$$\frac{d^2}{d\theta^2}\,\mathbb{E}\left[\varphi(B_\theta+K_\theta)\right] \;=\; -\,k_\theta\,\mathbb{E}\left[\varphi'(B_\theta+K_\theta)\right]$$

$$+\,(K'_\theta)^2\,\mathbb{E}\left[\varphi''(B_\theta+K_\theta)\right] +\, K'_\theta\,\mathbb{E}\left[\varphi'''(B_\theta+K_\theta)\right] +\, \frac{1}{4}\,\mathbb{E}\left[\varphi''''(B_\theta+K_\theta)\right].$$

Applying the integration by parts formulae:

$$\theta\int_{\mathbb{R}}\psi'(y+K_\theta)\,\mathcal{N}(0,\theta)(dy) \;=\; \int_{\mathbb{R}} y\,\psi(y+K_\theta)\,\mathcal{N}(0,\theta)(dy),$$

$$\theta^2\int_{\mathbb{R}}\psi''(y+K_\theta)\,\mathcal{N}(0,\theta)(dy) \;=\; \int_{\mathbb{R}}(y^2-\theta)\,\psi(y+K_\theta)\,\mathcal{N}(0,\theta)(dy),$$

to $\psi=\varphi''$, we obtain (3.7). $\square$

Proof of Proposition 3.2. Let $h\in C^2_c(0,1)$. By a density argument we can reduce to the case $\varphi\in C^2_b(\mathbb{R})$. Multiplying (3.7) by h_θ and integrating in θ we have, recalling (3.3):

$$\int_0^1 h_\theta\,k_\theta\,\mathbb{E}\left[\varphi'(B_\theta+K_\theta)\right]\,d\theta \;=\; -\int_0^1 h''_\theta\;\mathbb{E}\left[\varphi(B_\theta+K_\theta)\right]\,d\theta$$

$$+\,\mathbb{E}\left[\int_0^1 h_\theta\,\lambda(\theta,K'_\theta,B_\theta)\,\varphi''(B_\theta+K_\theta)\,d\theta\right], \tag{3.8}$$

where λ is defined by (3.3). By (3.1), (3.6) and the occupation times formula (2.1) this yields:

$$\mathbb{E}\left[\partial_{h\,\varphi'(B)}\Psi_k(B)\right] \;=\; -\int_0^1 h''_\theta\;\mathbb{E}\left[\varphi(B_\theta)\,\Psi_k(B)\right]\,d\theta$$

$$+\,\int_{\mathbb{R}}\mathbb{E}\left[\Psi_k(B)\int_0^1 h_\theta\,\lambda(\theta,K'_\theta,a-K_\theta)\,dL^a_\theta\right]\varphi''(a)\,da. \tag{3.9}$$

Therefore we conclude by (3.5). $\square$

4 Dirichlet forms on the Wiener space

In this section we introduce infinite dimensional Sobolev spaces which we need as spaces of test functions. Since we consider vector fields $\mathcal{K}$ taking values in C or L rather than in the Cameron-Martin space H^1, then the Malliavin derivative is not the correct notion of gradient and we must introduce a different differential calculus on L.

For $F \in \mathrm{Exp}(C)$, the usual derivative operator in the Malliavin calculus is $DF : C \mapsto L$, defined as follows:

$$\langle DF(\omega), \ell' \rangle \;:=\; \frac{d}{d\epsilon} F(\omega + \epsilon \ell)\bigg|_{\epsilon=0}, \qquad \ell \in H^1,$$

see e.g. §1.2 of [8]. Moreover we have closability in $L^2(\mu)$ of:

$$\mathcal{D}(F,F) \;:=\; \frac{1}{2}\,\mathbb{E}\left[\|DF(B)\|^2\right], \qquad F \in \mathrm{Dom}(D) \;=\; \mathrm{Dom}(\mathcal{D}),$$

and $\mathcal{D}$ is a Dirichlet form on the Wiener space. Then all functions in $\mathrm{Dom}(\mathcal{D})$ are differentiable in a weak sense along H^1-valued vector fields.

On the other hand we want to study $\partial_{h\,\varphi'(\omega)} F(\omega)$, see the l.h.s. of (2.3), and in general the regularity of $\theta \mapsto h_\theta\,\varphi'(\omega_\theta)$ is not better than that of $\omega \in C$. In particular the vector field $\mathcal{K}(\omega) := h\,\varphi'(\omega)$ is not H^1-valued and a general $F \in \mathrm{Dom}(\mathcal{D})$ can not be differentiated along $\mathcal{K}$.

For this reason we must consider here a different gradient $\nabla F : C \mapsto L = L^2(0,1)$ of $F \in \mathrm{Exp}(C)$, defined by:

$$\langle \nabla F(\omega), \ell \rangle \;:=\; \frac{d}{d\epsilon} F(\omega + \epsilon \ell)\bigg|_{\epsilon=0}, \qquad \ell \in L,$$

i.e. ∇F is the Fréchet differential of F in L. Also in this case we have closability in $L^2(\mu)$ of

$$\mathcal{E}(F,F) \;:=\; \frac{1}{2}\,\mathbb{E}\left[\|\nabla F(B)\|^2\right], \qquad F \in \mathrm{Dom}(\nabla) \;=\; \mathrm{Dom}(\mathcal{E}),$$

and $\mathcal{E}$ is a Dirichlet form on the Wiener space. Comparing the definitions of DF and ∇F we obtain $D = \mathcal{P}\nabla$ for all $F \in \mathrm{Exp}(C)$, where:

$$\mathcal{P} : L \mapsto L, \quad \mathcal{P}\ell_\theta \;:=\; \int_\theta^1 \ell_\tau\, d\tau, \quad \theta \in [0,1].$$

In particular for some constant $\kappa > 0$:

$$\mathcal{E}(F, F) \geq \kappa \mathcal{D}(F, F), \qquad \forall F \in \mathrm{Dom}(\mathcal{E}) \subset \mathrm{Dom}(\mathcal{D}).$$

For a discussion of these infinite dimensional Sobolev spaces, we refer to §9.2.1 for $\mathrm{Dom}(\mathcal{E})$ and to §9.3 for $\mathrm{Dom}(\mathcal{D})$ in [3]. We recall in particular that $\mathrm{Dom}(\mathcal{E})$ also admits a description in term of the Itô-Wiener decomposition: see e.g. Theorem 9.2.12 in [3].

Now all functions in $\mathrm{Dom}(\mathcal{E})$ can be differentiated, at least in a weak sense, along vector fields taking values in L or C, in particular along $\mathcal{K}(\omega) = h\,\varphi'(\omega)$. Moreover for $h \in C$ and $\varphi \in \mathrm{Conv}(\mathbb{R})$, setting:

$$\Phi_{h,\varphi} = \Phi : C \mapsto \mathbb{R}, \qquad \Phi(\omega) := \langle h, \varphi(\omega) \rangle = \int_0^1 h_\theta\, \varphi(\omega_\theta)\, d\theta,$$

then $\Phi \in \mathrm{Dom}(\mathcal{E})$ and $\nabla \Phi(\omega) = h\,\varphi'(\omega)$, i.e. for all $\omega \in C$:

$$\langle \nabla \Phi(\omega), \ell \rangle = \int_0^1 h_\theta\, \varphi'(\omega_\theta)\, \ell_\theta\, d\theta.$$

Then for all $F \in C^1_e(L)$ the l.h.s. of (2.3) is:

$$\mathbb{E}\left[\partial_{h\,\varphi'(B)} F(B)\right] = \mathbb{E}\left[\langle \nabla F(B), h\,\varphi'(B) \rangle\right] = 2\,\mathcal{E}(F, \Phi_{h,\varphi}). \tag{4.1}$$

We recall now that the semigroup $(P^{\mathcal{D}}_t : t \geq 0)$ in $L^2(\mu)$ associated with $\mathcal{D}$ is given by the Mehler formula:

$$P^{\mathcal{D}}_t F(z) = \int F(y)\, \mathcal{N}\left(e^{-t/2}\, z, (1 - e^{-t})\, Q\right)(dy), \qquad z \in C, \ F \in L^2(\mu),$$

where $\mathcal{N}(a, \mathcal{Q})$ denotes the Gaussian measure over L with mean $a \in L$ and covariance operator $\mathcal{Q} : L \mapsto L$. This semigroup is a basic tool in the Malliavin calculus: see e.g. Chapters 1-2 in [7] and §1.4-1.5 in [8].

Since in this paper we work with ∇ rather than with D, a crucial role is played by the transition semigroup $(P_t : t \geq 0)$ in $L^2(\mu)$ associated with $\mathcal{E}$, given by:

$$P_t F(z) = \int F(y)\, \mathcal{N}\left(e^{tA}\, z,\, Q_t\right)(dy), \qquad z \in C, \ F \in L^2(\mu),$$

where $(e^{tA} : t \geq 0)$ is the semigroup in L generated by the operator:

$$D(A) := \{h \in C : h'' \in L, \ h(0) = h'(1) = 0\}, \qquad Ah := \frac{1}{2}\, h'',$$

and we set:

$$Q_t := \int_0^t e^{2sA}\, ds = \frac{I - e^{2tA}}{-2A}, \qquad t \in [0,\infty]. \tag{4.2}$$

Notice in particular that:

$$Q_\infty = (-2A)^{-1} = Q. \tag{4.3}$$

The second equality of (4.3) says that Q and $-2A$ are inverse of one another and can be verified by an explicit computation.

The operators $(P_t^{\mathcal{D}} : t \geq 0)$ and $(P_t : t \geq 0)$ are two different examples of Ornstein-Uhlenbeck semigroups: we refer to Chapters 6 and 10 in [3]. For a more detailed description of $(P_t : t \geq 0)$ see section 6 below.

Two important properties of $\mathrm{Dom}(\mathcal{E})$ are stated in the following:

Lemma 4.1. *The space* $\mathrm{Lip}_e(L)$ *is contained in* $\mathrm{Dom}(\mathcal{E})$. *The space* $\mathrm{Exp}(C)$ *is dense in* $\mathrm{Dom}(\mathcal{E})$.

Proof. We recall that $F \in \mathrm{Dom}(\mathcal{E})$ if and only if $\sup_{t>0} \mathcal{E}(P_t F, P_t F) < \infty$. Now:

$$|P_t F(z_1) - P_t F(z_2)| \leq \int |F(y + e^{tA} z_1) - F(y + e^{tA} z_2)| \, \mathcal{N}(0, Q_t)(dy)$$

$$\leq \int e^{c(\|y\| + \|z_1\|)} \, \|z_1 - z_2\| \, \mathcal{N}(0, Q_t)(dy) \leq \kappa \, e^{c\|z_1\|} \, \|z_1 - z_2\|,$$

so that $\|\nabla P_t F(z)\| \leq e^{c\|z\|}$ for all $z \in C$ and we obtain the first claim. For the second one, we refer to §9.2.1 of [3]. $\square$

5 Proof of the main results

We want to use the tools introduced in the previous section to prove Theorems 2.1, 2.2 and 2.3.

In Propositions 3.1 and 3.2 we have proved that (2.2) and (2.3) hold for all $F \in \mathrm{Exp}(C)$. This space is dense in the topology of the Sobolev space $\mathrm{Dom}(\mathcal{E})$, introduced in the previous section. An a priori estimate, given in Lemma 5.3, and a density argument allow to extend (2.2) and (2.3) to much larger spaces of test functions and to prove Theorems 2.1 and 2.2 and also Theorem 2.3 as a corollary. In particular, in this section we prove:

Proposition 5.1. *Let $h \in C_c(0,1)$ and $a \in \mathbb{R}$. Then the limit in (2.2) exists for all $F \in \mathrm{Lip}_e(L)$.*

Proposition 5.2. *For all $h \in h_c^2(0,1)$, $\varphi \in \mathrm{Conv}(\mathbb{R})$ and $F \in \mathrm{Lip}_e(L)$:*

$$\begin{aligned}\mathbb{E}\left[\langle \nabla F(B), h\,\varphi'(B)\rangle\right] \ &= \ -\,\mathbb{E}\left[F(B)\int_0^1 h''_\theta\,\varphi(B_\theta)\,d\theta\right] \\ &\quad + \int_{\mathbb{R}} \mathbb{E}\left[F(B)\int_0^1 h_\theta\ :\dot{B}^2_\theta:\ dL^a_\theta\right]\varphi''(da).\end{aligned} \tag{5.1}$$

Proposition 5.1 proves Theorem 2.1. Theorem 2.2 follows by Proposition 5.2 and formula (4.1), recalling that $C^1_e(L) \subset \mathrm{Lip}_e(L)$. At the end of the section, we derive Theorem 2.3 from Proposition 5.2. We also recall that ∇F is well defined, since by Lemma 4.1: $F \in \mathrm{Lip}_e(L) \subset \mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\nabla)$.

We recall that μ denotes the Wiener measure, law of B, i.e. for all bounded Borel $F : C \mapsto \mathbb{R}$:

$$\mu(F) \ = \int F\,d\mu \ = \ \mathbb{E}[F(B)].$$

By Proposition 10.5.2 of [3], $\mathcal{E}$ satisfies the Poincaré inequality:

$$\int (F-\mu(F))^2\,d\mu \ \le \ \frac{1}{\lambda_1}\,\mathcal{E}(F,F), \qquad F \in \mathrm{Dom}(\mathcal{E}),$$

where $\lambda_1 = \pi^2/4$, see (6.7) below. Since $(P_t : t \ge 0)$ is the semigroup in $L^2(\mu)$ associated with $\mathcal{E}$, the Poincaré inequality implies the exponential convergence of P_tF to $\mu(F)$ in $L^2(\mu)$:

$$\|P_tF - \mu(F)\|^2_{L^2(\mu)} \ \le \ e^{-2t/\lambda_1}\,\|F\|^2_{L^2(\mu)}, \qquad t \ge 0,\ F \in L^2(\mu). \tag{5.2}$$

In particular for all $G \in L^2(\mu)$:

$$\mathcal{R}G \ := \int_0^\infty (P_tG - \mu(G))\,dt \ \in \ \mathrm{Dom}(\mathcal{E}),$$

and for all $F \in \mathrm{Dom}(\mathcal{E})$:

$$\mathbb{E}[F(B)\,G(B)] \ = \ \mathbb{E}[F(B)]\,\mathbb{E}[G(B)] \ + \ \mathcal{E}(F, \mathcal{R}G).$$

Let now $h \in C_c(0,1)$ and $a \in \mathbb{R}$. For all $\epsilon > 0$ we define $G_{\epsilon,a} \in L^2(\mu)$

$$G_{\epsilon,a}(B) \ := \int_0^1 h_\theta\ :\dot{B}^2_{\epsilon,\theta}:\ dL^a_\theta, \qquad G_\epsilon \ := \ G_{\epsilon,0}. \tag{5.3}$$

Then (2.2) is equivalent to the existence of the limit as $\epsilon \to 0$ of:

$$\mathbb{E}\left[F(B)\, G_{\epsilon,a}(B)\right] \,=\, \mathbb{E}[F(B)]\,\mathbb{E}[G_{\epsilon,a}(B)] \,+\, \mathcal{E}(F, \mathcal{R}G_{\epsilon,a}) \tag{5.4}$$

for all $F \in \mathrm{Lip}_e(L)$. The main tool in the proof of Propositions 5.1 and 5.2 is the following estimate:

Lemma 5.3. *If $h \in C_c(0,1)$ then there exists a constant $\kappa > 0$ such that:*

$$\|P_t G_\epsilon\|^2_{L^2(\mu)} \,\leq\, \kappa\, \frac{1 + |\ln t|^6}{t^{3/4}}, \qquad t \in (0,1],\ \epsilon > 0. \tag{5.5}$$

The proof of Lemma 5.3 is postponed to section 6. As a consequence of Lemma 5.3 we have the following:

Proposition 5.4. *Let $h \in C_c(0,1)$ and $a = 0$. Then the limit in (5.4) exists for all $F \in \mathrm{Dom}(\mathcal{E})$.*

Proof. By Lemma 3.3 for $k = 0$ and $\psi \in C_b(\mathbb{R})$ we have:

$$\mathbb{E}\left[\int_0^1 h_\theta \ :\dot{B}^2_{\epsilon,\theta}:\ \psi(B_\theta)\, d\theta\right] \,=\, \mathbb{E}\left[\int_0^1 h_\theta\, \frac{B_\theta^2 - \theta}{4\theta^2}\, \psi(B_\theta)\, d\theta\right].$$

By the occupation times formula (2.1) we obtain for all $\psi \in C_b(\mathbb{R})$:

$$\int_{\mathbb{R}} \mathbb{E}[G_{\epsilon,a}(B)]\, \psi(a)\, da \,=\, \int_0^1 h_\theta \int_{\mathbb{R}} \frac{a^2 - \theta}{4\theta^2}\, \frac{e^{-a^2/2\theta}}{\sqrt{2\pi\theta}}\, \psi(a)\, da\, d\theta.$$

In particular:

$$\mathbb{E}[G_{\epsilon,a}(B)] \,=\, \int_0^1 h_\theta\, \frac{a^2 - \theta}{4\theta^2}\, \frac{e^{-a^2/2\theta}}{\sqrt{2\pi\theta}}\, d\theta,$$

which does not depend on ϵ. Therefore, by (5.4) the existence of the limit in (2.2) with $a = 0$ for all $F \in \mathrm{Dom}(\mathcal{E})$ is equivalent to the weak convergence of $\mathcal{R}G_\epsilon$ in $\mathrm{Dom}(\mathcal{E})$. Now, by Proposition 3.1, the limit in (2.2) with $a = 0$ exists for all $F \in \mathrm{Exp}(C)$, which is dense in $\mathrm{Dom}(\mathcal{E})$. Therefore, if we can prove that:

$$\sup_{\epsilon > 0}\, \mathcal{E}(\mathcal{R}G_\epsilon, \mathcal{R}G_\epsilon) \,<\, \infty, \tag{5.6}$$

then we conclude. Indeed, for any $F \in \mathrm{Dom}(\mathcal{E})$ we can find a sequence $(F_n)_n \subset \mathrm{Exp}(C)$ converging to F in $\mathrm{Dom}(\mathcal{E})$. Write:

$$|\mathcal{E}(F, G_\epsilon - G_\delta)| \,\leq\, |\mathcal{E}(F_n, G_\epsilon - G_\delta)| \,+\, |\mathcal{E}(F - F_n, G_\epsilon - G_\delta)|.$$

By (5.6) we can make the second term arbitrarily small for some n big enough but fixed, uniformly in $\epsilon, \delta > 0$. Then by Proposition 3.1 we can make the first term arbitrarily small as $\epsilon, \delta \to 0$.

For the proof of (5.6), we recall the following formula:

$$
\begin{aligned}
\mathcal{E}(\mathcal{R}G_\epsilon, \mathcal{R}G_\epsilon) &= \int \mathcal{R}G_\epsilon \, (G_\epsilon - \mu(G_\epsilon)) \, d\mu \\
&= \int_0^\infty \int (P_t G_\epsilon - \mu(G_\epsilon)) \, (G_\epsilon - \mu(G_\epsilon)) \, d\mu \, dt \\
&= \int_0^\infty \|P_{t/2} G_\epsilon - \mu(G_\epsilon)\|^2_{L^2(\mu)} \, dt.
\end{aligned}
$$

Moreover by (5.2) and (5.5), since $P_{1+t} = P_t P_1$, $t \geq 0$:

$$
\|P_{1+t} G_\epsilon - \mu(G_\epsilon)\|^2_{L^2(\mu)} \leq e^{-2t/\lambda_1} \|P_1 G_\epsilon\|^2_{L^2(\mu)} \leq \kappa \, e^{-2t/\lambda_1}.
$$

Therefore (5.6) follows from:

$$
\begin{aligned}
\mathcal{E}(\mathcal{R}G_\epsilon, \mathcal{R}G_\epsilon) &\leq \int_0^1 \|P_{t/2} G_\epsilon\|^2_{L^2(\mu)} \, dt + \int_1^\infty \|P_{t/2} G_\epsilon - \mu(G_\epsilon)\|^2_{L^2(\mu)} \, dt \\
&\leq \kappa \int_0^1 \frac{1 + |\ln t|^6}{t^{3/4}} \, dt + \kappa \int_1^\infty e^{-2t/\lambda_1} \, dt < \infty. \quad \square
\end{aligned}
$$

We can now apply the results of Propositions 3.1, 3.2 and 5.4 to prove Propositions 5.1 and 5.2.

Proof of Proposition 5.1. We fix $\delta \in (0, 1/2)$ such that $\mathrm{supp}(h) \subset [\delta, 1-\delta]$ and we consider $\epsilon \in (0, \delta/2)$. By Proposition 5.4, (2.2) holds for $a = 0$ and for all $F \in \mathrm{Dom}(\mathcal{E})$.

Let $\ell : [0,1] \mapsto \mathbb{R}$ be of class C^2 such that $\ell_0 = 0$ and $\ell_\theta = 1$ for all $\theta \in [\delta/2, 1]$. By the Cameron-Martin theorem we have the following formula:

$$
\mathbb{E}\left[F(B)\right] = \mathbb{E}\left[F(B + a\ell) \, \exp\left(a \langle \ell'', B \rangle - c(\ell, a)\right)\right], \tag{5.7}
$$

where $c(\ell, a) := a^2 \|\ell'\|^2 / 2$. If $G_{\epsilon,a}$ is defined as in (5.3), then almost surely:

$$
G_{\epsilon,a}(B + a\ell) = G_{\epsilon,a}(B + a) = G_{\epsilon,0}(B) = G_\epsilon(B),
$$

where the first equality holds because h vanishes where $\ell \neq 1$ and the second one because the local time of $B + a$ at a is equal to the local time of B at 0. Let now F be in $\mathrm{Lip}_e(L)$. Then by (5.7):

$$
\mathbb{E}\left[F(B) \, G_{\epsilon,a}(B)\right] = \mathbb{E}\left[F_a(B) \, G_\epsilon(B)\right], \tag{5.8}
$$

where : $\qquad F_a(z) \ := \ F(z+a\ell) \, \exp(a\langle \ell'', z\rangle - c(\ell,a)), \quad z \in C.$

Now, $F_a \in \mathrm{Dom}(\mathcal{E})$, so that, by Proposition 5.4, $\mathbb{E}[F(B)\, G_{\epsilon,a}(B)]$ converges as $\epsilon \to 0$ and (2.2) is proven for all $a \in \mathbb{R}$. $\quad\square$

Proof of Proposition 5.2. We consider first the case:

$$\varphi(x) \ := \ |x-a| \ \Longrightarrow \ \varphi'(x) \ = \ \mathrm{sign}(x-a), \quad \varphi''(dx) \ = \ 2\,\delta_a(dx),$$

for some $a \in \mathbb{R}$, where δ_a is the Dirac mass at a and

$$\mathrm{sign} : \mathbb{R} \mapsto \{0,1\}, \qquad \mathrm{sign}(x) \ := \ 1_{(0,\infty)}(x) \ - \ 1_{(-\infty,0]}(x).$$

In this case, (5.1) becomes:

$$\begin{aligned} \mathbb{E}\left[\langle \nabla F(B), h\,\mathrm{sign}(B-a)\rangle\right] \ &= \ -\,\mathbb{E}\left[F(B) \int_0^1 h''_\theta \, |B_\theta - a| \, d\theta\right] \qquad (5.9) \\ &\quad + 2\,\mathbb{E}\left[F(B) \int_0^1 h_\theta \ :\dot{B}^2_\theta: \ dL^a_\theta\right]. \end{aligned}$$

Consider first the case $a = 0$. By Proposition 5.4, the r.h.s. of (5.9) defines a bounded linear functional on $\mathrm{Dom}(\mathcal{E})$. Moreover, by Proposition 3.1, (5.9) holds for all $F \in \mathrm{Exp}(C)$. Since both sides of (5.9) are bounded linear functionals on $\mathrm{Dom}(\mathcal{E})$, coinciding on the dense subset $\mathrm{Exp}(C)$, they coincide on $\mathrm{Dom}(\mathcal{E})$. Therefore (5.9) is proven for $a = 0$.

Let ℓ and F_a be the functions introduced in the proof of Proposition 5.1. By (5.8) and by (5.9) with $a = 0$ we obtain:

$$\begin{aligned} &\lim_{\epsilon\to 0} 2\,\mathbb{E}\left[F(B)\, G_{\epsilon,a}(B)\right] \ = \ \lim_{\epsilon\to 0} 2\,\mathbb{E}\left[F_a(B)\, G_\epsilon(B)\right] \\ &= \ \mathbb{E}\left[\langle \nabla F_a(B), h\,\mathrm{sign}(B)\rangle\right] + \mathbb{E}\left[F_a(B) \int_0^1 h''_\theta \, |B_\theta| \, d\theta\right] \\ &= \ \mathbb{E}\left[\langle \nabla F(B), h\,\mathrm{sign}(B-a)\rangle\right] + \mathbb{E}\left[F(B) \int_0^1 h''_\theta \, |B_\theta - a| \, d\theta\right]. \end{aligned}$$

Therefore (5.9) is proven for all $a \in \mathbb{R}$ and $F \in \mathrm{Lip}_e(L)$. Let now $\varphi \in C^2_c(\mathbb{R})$. Multiplying (5.9) by $\varphi''(a)$ and integrating in da we obtain (5.1) and:

$$\begin{aligned} &\left| \int_{\mathbb{R}} \mathbb{E}\left[F(B) \int_0^1 h_\theta \ :\dot{B}^2_\theta: \ dL^a_\theta\right] \varphi''(a)\, da \right| \\ &\leq \ \kappa \int_0^1 \mathbb{E}\left[e^{c\|B\|} \left(|\varphi(B_\theta)| + |\varphi'(B_\theta)|\right)\right] d\theta. \end{aligned}$$

Therefore, by a density argument (5.1) holds for all $\varphi \in \mathrm{Conv}(\mathbb{R})$. $\square$

Proof of Theorem 2.3. We start by recalling that, by Tanaka's formula, $\ell^0 \equiv 2\, L^a$, where ℓ^0 is the local time process of $X = |B - a|$ at 0 and L^a is the local time process of B at a.

Fix $h \in C^2_c(0,1)$ and $f \in C^1_e(L)$. Setting $F(z) := f(|z-a|)$, $z \in L$, then clearly $F \in \mathrm{Lip}_e(L)$. By Lemma 4.1, $F \in \mathrm{Dom}(\mathcal{E})$ and by the chain rule:

$$\langle \nabla F(z), h \rangle \, = \, \langle \nabla f(|z-a|), h \,\mathrm{sign}(z-a) \rangle, \qquad \mu - \text{a.e. } z.$$

In particular for μ-a.e. z:

$$\langle \nabla F(z), h \,\mathrm{sign}(z-a) \rangle \, = \, \langle \nabla f(|z-a|), h \rangle,$$

since $[\mathrm{sign}(z-a)]^2 \equiv 1$. Therefore, formula (5.9) applied to $F(z) := f(|z-a|)$ and $\varphi(x) = |x-a|$, $z \in L$, $x \in \mathbb{R}$, yields (2.4). $\square$

6 The main estimate

In this section we prove Lemma 5.3. We recall that G_ϵ is the sum of two diverging terms. Applying P_t to G_ϵ we have a regularization effect: indeed, we write $P_t G_\epsilon$ as a sum of terms, which after some cancelations converge as ϵ tends to 0. This compensation of infinities requires a careful study of each term.

We start with a more detailed description of the semigroup $(P_t : t \geq 0)$ of the Dirichlet Form $\mathcal{E}$ in $L^2(\mu)$, defined in section 4. We introduce first the Green function $(g_t(\theta,\theta') : t > 0, \theta, \theta' \in [0,1])$ of the heat equation associated with A, i.e. solution of

$$\frac{\partial g}{\partial t} \, = \, \frac{1}{2}\frac{\partial^2 g}{\partial \theta^2}$$

with boundary and initial conditions:

$$g_t(0,\theta') \, = \, \frac{\partial g_t}{\partial \theta}(1,\theta') = 0, \qquad g_0(\theta,\theta') \, = \, \delta_\theta(d\theta'),$$

where δ_θ is the Dirac mass at θ. Then we set for all $z \in C$:

$$z(t,\theta) \, := \int_0^1 g_t(\theta,\theta')\, z_{\theta'}\, d\theta', \quad v(t,\theta) \, := \int_0^t \int_0^1 g_{t-s}(\theta,\theta')\, W(d\theta', ds), \quad (6.1)$$

$$u(t,\theta) := z(t,\theta) + v(t,\theta), \qquad U_t(z) := u(t,\cdot) \in C, \tag{6.2}$$

where $(W(\theta', s) : \theta' \in [0,1], s \geq 0)$ is a Brownian sheet. Then:

$$P_t F(z) = \mathbb{E}\left[F(U_t(z))\right], \qquad t \geq 0, \; z \in C, \; F \in L^2(\mu).$$

Although this is not needed in this paper, we remark that $(u(t,\theta) : t \geq 0, \theta \in [0,1])$ is the unique solution of the Stochastic Partial Differential Equation driven by space-time white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 W}{\partial t \partial \theta} \\[2ex] u(t,0) = \dfrac{\partial u}{\partial \theta}(t,1) = 0 \\[2ex] u(0,\theta) = z_\theta, \end{cases}$$

see [11].

Notice that $(z(t,\theta) : t \geq 0, \theta \in [0,1])$ is a deterministic continuous function and $(v(t,\theta) : t \geq 0, \theta \in [0,1])$ is a centered continuous Gaussian process. A crucial role is played by the function:

$$q_t(\theta,\theta') := \mathbb{E}\left[v(t,\theta)\, v(t,\theta')\right] = \int_0^t g_{2s}(\theta,\theta')\, ds, \qquad q_t(\theta) := q_t(\theta,\theta), \tag{6.3}$$

for $\theta, \theta' \in [0,1]$, $t \geq 0$. Notice that for all $\ell \in L$:

$$\int_0^1 q_t(\theta,\theta')\, \ell_{\theta'}\, d\theta' = \int_0^t e^{2sA} \ell_\theta\, ds = Q_t \ell_\theta,$$

where Q_t is defined in (4.2). By (4.3) above, $Q_\infty = Q$, i.e.

$$q_\infty(\theta,\theta') := \lim_{t \nearrow \infty} q_t(\theta,\theta') = \theta \wedge \theta', \qquad q_\infty(\theta) := q_\infty(\theta,\theta) = \theta. \tag{6.4}$$

We also set:

$$q^t(\theta,\theta') := [q_\infty - q_t](\theta,\theta') = \int_t^\infty g_{2s}(\theta,\theta')\, ds, \qquad q^t(\theta) := q^t(\theta,\theta). \tag{6.5}$$

We denote by $\gamma_t(\theta - \theta')$ the density of the Gaussian measure $\mathcal{N}(\theta,t)(d\theta')$ over $\mathbb{R}$ with mean θ and variance t. Then $g - \gamma$ is smooth over $[0,\infty) \times$

$(0,1)\times(0,1)$. In particular for all $\delta\in(0,1/2)$ there exists a constant $\kappa_\delta>0$ such that for all $t\in[0,1]$, $\theta\in[\delta,1-\delta]$:

$$q_t(\theta) \,=\, \int_0^t \frac{ds}{\sqrt{4\pi s}} \,+\, \int_0^t (g_{2s}(\theta,\theta)-\gamma_{2s}(0))\,ds \,\geq\, \kappa_\delta\, t^{1/2}. \tag{6.6}$$

Finally, we introduce the complete orthonormal system of L:

$$e_i(\theta) \,:=\, 2^{1/2}\,\sin\left(\sqrt{\lambda_i}\,\theta\right), \quad \theta\in[0,1], \qquad \lambda_i \,:=\, \frac{\pi^2}{4}\,(2i-1)^2,$$

$i=1,2,\ldots$. Then $(e_i)_i$ is a system of eigenvectors of Q, A and e^{tA}:

$$Q\,e_i \,=\, \frac{1}{\lambda_i}\,e_i, \qquad A\,e_i \,=\, -\,\frac{\lambda_i}{2}\,e_i, \qquad e^{tA}\,e_i \,=\, e^{-t\lambda_i/2}\,e_i. \tag{6.7}$$

In particular:

$$q_t(\theta,\theta') \,=\, \sum_{i=1}^{\infty} \frac{1-e^{-\lambda_i t}}{\lambda_i}\,e_i(\theta)\,e_i(\theta'), \qquad t\in[0,\infty],\ \theta,\theta'\in[0,1]. \tag{6.8}$$

Proof of Lemma 5.3. We fix $\delta\in(0,1/2)$ such that $\mathrm{supp}(h)\subseteq[\delta,1-\delta]$ and we consider $\epsilon\in(0,\delta)$. Recalling (6.1) and (6.2), we set

$$v_\epsilon(t,\cdot) \,:=\, \rho_\epsilon * v(t,\cdot), \quad z_\epsilon(t,\cdot) \,:=\, \rho_\epsilon * z(t,\cdot), \quad u_\epsilon \,:=\, z_\epsilon + v_\epsilon.$$

We denote the partial derivative w.r.t. θ by ∂_θ.

An explicit formula for P_tG_ϵ. By the definition (5.3) of $G_{\epsilon,a}$ and by the occupation times formula, for μ-a.e. ω:

$$\int_{\mathbb{R}} G_{\epsilon,a}(\omega)\,\psi(a)\,da \,=\, \int_0^1 h_\theta\,\left((\omega'_{\epsilon,\theta})^2 - c_{\epsilon,\theta}\right)\,\psi(\omega_\theta)\,d\theta,$$

for any $\psi\in C_b(\mathbb{R})$. By Fubini's theorem:

$$\begin{aligned} \int_{\mathbb{R}} P_tG_{\epsilon,a}(z)\,\psi(a)\,da \;&=\; P_t\left[\int_{\mathbb{R}} G_{\epsilon,a}\,\psi(a)\,da\right](z) \\ &=\; \int_0^1 h_\theta\,\mathbb{E}\left[\psi(u(t,\theta))\left((\partial_\theta u_\epsilon(t,\theta))^2 - c_{\epsilon,\theta}\right)\right]\,d\theta. \end{aligned} \tag{6.9}$$

As in the proof of Lemma 3.3 we set for fixed $t > 0$ and $\theta \in (0,1)$:

$$\ell_\sigma := \frac{q_t(\sigma,\theta)}{q_t(\theta)}, \qquad \hat{v}(t,\sigma) := v(t,\sigma) - v(t,\theta)\,\ell_\sigma, \qquad \sigma \in (0,1).$$

Then the covariance between the two Gaussian variables $\hat{v}(t,\cdot)$ and $v(t,\theta)$ is zero, so that $\hat{v}(t,\cdot)$ and $v(t,\theta)$ are independent. Denoting $\overline{z} := z(t,\theta)$ and $\overline{q} := q_t(\theta)$ we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\psi(u(t,\theta))\left[(\partial_\theta u_\epsilon(t,\theta))^2 - c_{\epsilon,\theta}\right]\right] \qquad (6.10)\\
&= \int_{\mathbb{R}} \mathcal{N}(0,\overline{q})(dy)\ \psi(y+\overline{z})\,\mathbb{E}\left[\left(\partial_\theta u_\epsilon(t,\theta) + (y - v(t,\theta))\,\ell'_{\epsilon,\theta}\right)^2 - c_{\epsilon,\theta}\right]\\
&= \int_{\mathbb{R}} \mathcal{N}(0,\overline{q})(dy)\,\psi(y+\overline{z})\ \left[\left(\partial_\theta z_\epsilon(t,\theta) +\, y\,\ell'_{\epsilon,\theta}\right)^2 - \overline{q}\left(\ell'_{\epsilon,\theta}\right)^2 - c^t_{\epsilon,\theta}\right]
\end{aligned}$$

where, recalling (4.2) and setting $Q^t := e^{tA}Qe^{tA} = Q - Q_t$, by (3.2):

$$c^t_{\epsilon,\theta} := c_{\epsilon,\theta} - \mathbb{E}\left[(\partial_\theta v_\epsilon(t,\theta))^2\right] = \langle Q^t\rho'_\epsilon(\cdot-\theta), \rho'_\epsilon(\cdot-\theta)\rangle. \qquad (6.11)$$

Therefore, by (6.9) and (6.10):

$$\begin{aligned}
\int_{\mathbb{R}} P_t G_{\epsilon,a}(z)\,\psi(a)\,da &= \int_0^1 d\theta\,h_\theta \int_{\mathbb{R}} \mathcal{N}(0,q_t(\theta))(dy)\,\psi(y+z(t,\theta))\ \cdot\\
&\cdot \left[(\partial_\theta z_\epsilon(t,\theta))^2 - c^t_{\epsilon,\theta} + 2\,y\,\ell'_{\epsilon,\theta}\,\partial_\theta z_\epsilon(t,\theta) + \left(y^2 - q_t(\theta)\right)\left(\ell'_{\epsilon,\theta}\right)^2\right].
\end{aligned}$$

Therefore we obtain:

$$\begin{aligned}
P_t G_\epsilon(z) = \int_0^1 h_\theta\,\frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}}&\left[(\partial_\theta z_\epsilon(t,\theta))^2 - c^t_{\epsilon,\theta}\right.\\
&\left. -2\,z(t,\theta)\,\ell'_{\epsilon,\theta}\,\partial_\theta z_\epsilon(t,\theta) + \left((z(t,\theta))^2 - q_t(\theta)\right)\left(\ell'_{\epsilon,\theta}\right)^2\right] d\theta,
\end{aligned}$$

and

$$\|P_t\,G_\epsilon\|^2 \le 4\sum_{i=1}^3 I_i(t,\epsilon), \qquad I_i(t,\epsilon) := \|V^i_{\epsilon,t}\|^2,$$

where

$$
\begin{aligned}
V^1_{\epsilon,t}(z) &:= \int_0^1 h_\theta \, \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \left[(\partial_\theta z_\epsilon(t,\theta))^2 - c^t_{\epsilon,\theta} \right] d\theta, \\
V^2_{\epsilon,t}(z) &:= -\int_0^1 h_\theta \, \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \, 2\, z(t,\theta)\, \ell'_{\epsilon,\theta}\, \partial_\theta z_\epsilon(t,\theta)\, d\theta, \\
V^3_{\epsilon,t}(z) &:= \int_0^1 h_\theta \, \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \, \left[(z(t,\theta))^2 - q_t(\theta) \right] \left(\ell'_{\epsilon,\theta} \right)^2 \, d\theta.
\end{aligned}
$$

For $F \in C^1_e(L)$, $k \in L$ and $K := Qk \in H^1$ we have integrating by parts w.r.t. the Wiener measure:

$$
\mathbb{E}\left[\partial_K F(B)\right] \, = \, \mathbb{E}\left[F(B) \int_0^1 K'_\theta \, dB_\theta \right].
$$

On the other hand, integrating by parts on $[0,1]$ we obtain:

$$
\int_0^1 K'_\theta \, dB_\theta \, = \, K'_1 \, B_1 \, - \, K'_0 \, B_0 \, - \int_0^1 K''_\theta \, B_\theta \, d\theta \, = \, \int_0^1 k_\theta \, B_\theta \, d\theta,
$$

since $K'_1 = B_0 = 0$. Therefore we obtain the following formula:

$$
\mathbb{E}\left[F(B)\,\langle k, B\rangle\right] \, = \, \mathbb{E}\left[\partial_K F(B)\right]. \tag{6.12}
$$

Iterating (6.12) several times we obtain for $F \in C^4_b(L)$, $k^i \in L$ and $K^i := Qk^i$:

$$
\begin{aligned}
&\mathbb{E}\left[F(B)\,\langle k^1, B\rangle\,\langle k^2, B\rangle \right] \qquad (6.13) \\
&= \, \langle K^1, k^2\rangle\,\mathbb{E}\left[F(B)\right] \, + \, \mathbb{E}\left[\partial^2_{K^1,K^2} F(B) \right],
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[F(B)\,\langle k^1, B\rangle^2\,\langle k^2, B\rangle^2 \right] \qquad (6.14) \\
&= \, \left(\langle K^1, k^1\rangle\,\langle K^2, k^2\rangle \, + \, 2\,\langle K^1, k^2\rangle^2 \right) \mathbb{E}\left[F(B)\right] \\
&\quad + \sum_{i \neq j} \langle K^i, k^i\rangle\,\mathbb{E}\left[\partial^2_{K^j,K^j} F(B) \right] \, + \, 4\,\langle K^2, k^1\rangle\,\mathbb{E}\left[\partial^2_{K^1,K^2} F(B) \right] \\
&\quad + \mathbb{E}\left[\partial^4_{K^1,K^1,K^2,K^2} F(B) \right].
\end{aligned}
$$

Estimate of I_1. We set for the rest of the proof:

$$k^1 := -\, e^{tA}\, \rho'_\epsilon(\cdot - \theta), \qquad k^2 := -\, e^{tA}\, \rho'_\epsilon(\cdot - \theta'), \qquad K^i \,:=\, Qk^i, \qquad (6.15)$$

$$F^{a,b}(z) \,:=\, \frac{e^{-(z(t,\theta)-a)^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \cdot \frac{e^{-(z(t,\theta')-b)^2/2q_t(\theta')}}{\sqrt{2\pi q_t(\theta')}}, \qquad F \,:=\, F^{0,0},$$

for $z \in L$ and $a, b \in \mathbb{R}$. Then we have

$$I_1(t,\epsilon) \,=\, \int \mu(dz) \left[\int_0^1 h_\theta \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \,\left[(\partial_\theta z_\epsilon(t,\theta))^2 - c^t_{\epsilon,\theta}\right]\, d\theta\right]^2$$

$$= \int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'} \,\cdot$$

$$\cdot\, \mathbb{E}\left[F(B)\, \left(\langle k^1, B\rangle^2 \langle k^2, B\rangle^2 \,-\, \langle k^1, B\rangle^2\, c^t_{\epsilon,\theta'} \,-\, c^t_{\epsilon,\theta}\, \langle k^2, B\rangle^2 \,+\, c^t_{\epsilon,\theta}\, c^t_{\epsilon,\theta'}\right)\right].$$

Moreover by (3.2) and (6.11):

$$\langle K^1, k^1\rangle \,=\, \langle Q^t \rho'(\cdot - \theta), \rho'(\cdot - \theta)\rangle \,=\, c^t_{\epsilon,\theta}, \qquad \langle K^2, k^2\rangle \,=\, c^t_{\epsilon,\theta'},$$

$$\langle K^1, k^2\rangle \,=\, \langle Q^t \rho'(\cdot - \theta), \rho'(\cdot - \theta')\rangle \,=:\, c^t_{\epsilon,\theta,\theta'}.$$

Using (6.13) and (6.14), several terms cancel and what remains is:

$$I_1(t,\epsilon) \,=\, \int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'} \,\cdot$$

$$\cdot\, \mathbb{E}\left[F(B)\, 2\, \langle K^1, k^2\rangle^2 +\, 4\, \langle K^1, k^2\rangle\, \partial^2_{K^1,K^2} F(B) \,+\, \partial^4_{K^1,K^1,K^2,K^2} F(B)\right].$$

Notice that the function $\Gamma : \mathbb{R}^2 \mapsto \mathbb{R}_+$

$$\Gamma(a,b) \,:=\, \mathbb{E}\left[F^{a,b}(B)\right] \,=\, \mathbb{E}\left[\frac{\exp\left(-\frac{(\langle B, e^{tA}\delta_\theta\rangle - a)^2}{2q_t(\theta)} \,-\, \frac{(\langle B, e^{tA}\delta_{\theta'}\rangle - b)^2}{2q_t(\theta')}\right)}{2\pi\sqrt{q_t(\theta)\, q_t(\theta')}}\right]$$

is the density of the convolution between $\mathcal{N}(0, q_t(\theta)) \otimes \mathcal{N}(0, q_t(\theta'))$ and the law of $(\langle B, e^{tA}\delta_\theta\rangle, \langle B, e^{tA}\delta_{\theta'}\rangle)$. Therefore Γ is the density of the Gaussian measure on $\mathbb{R}^2$ with zero mean and covariance matrix:

$$\begin{pmatrix} q_t(\theta) & 0 \\ 0 & q_t(\theta') \end{pmatrix} \,+\, \begin{pmatrix} q^t(\theta) & q^t(\theta,\theta') \\ q^t(\theta,\theta') & q^t(\theta') \end{pmatrix} \,=\, \begin{pmatrix} q_\infty(\theta) & q^t(\theta,\theta') \\ q^t(\theta,\theta') & q_\infty(\theta') \end{pmatrix} =:\, \Lambda_{\theta,\theta'}.$$

Moreover

$$(q^t(\theta,\theta'))^2 \;=\; \left(\mathbb{E}\left[\langle B, e^{tA}\delta_\theta\rangle\,\langle B, e^{tA}\delta_{\theta'}\rangle\right]\right)^2$$

$$\le\; \mathbb{E}\left[\langle B, e^{tA}\delta_\theta\rangle^2\right]\mathbb{E}\left[\langle B, e^{tA}\delta_{\theta'}\rangle^2\right] \;=\; q^t(\theta)\,q^t(\theta') \;\le\; q^t(\theta)\,q_\infty(\theta').$$

Using this inequality and recalling that $q_\infty - q^t = q_t$ we have:

$$\det\Lambda_{\theta,\theta'} \;=\; q_\infty(\theta)\,q_\infty(\theta') - (q^t(\theta,\theta'))^2 \;\ge\; q_t(\theta)\,q_\infty(\theta').$$

Therefore by (6.6), for $\theta,\theta'\in[\delta,1-\delta]$:

$$\mathbb{E}\left[F(B)\right] \;=\; \Gamma(0,0) \;=\; \frac{1}{2\pi(\det\Lambda_{\theta,\theta'})^{1/2}} \;\le\; \frac{\kappa_\delta^{-1/2}}{t^{1/4}}.$$

Now, by (6.8):

$$c^t_{\epsilon,\theta,\theta'} \;=\; \langle Q^t\rho'_\epsilon(\cdot-\theta), \rho'_\epsilon(\cdot-\theta')\rangle \;=\; \sum_{i=1}^\infty \frac{e^{-\lambda_i t}}{\lambda_i}\,(\rho_\epsilon * e'_i)_\theta\,(\rho_\epsilon * e'_i)_{\theta'}.$$

Setting $\eta_i := \lambda_i^{1/2} e_i$ we have that $(\eta_i)_{i\in\mathbb{N}}$ is a c.o.s. in L. We obtain

$$\int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\,\mathbb{E}\left[F(B)\right]\left(c^t_{\epsilon,\theta,\theta'}\right)^2 \;\le\; \frac{\kappa}{t^{1/4}}\int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\;\left(c^t_{\epsilon,\theta,\theta'}\right)^2$$

$$=\; \frac{\kappa}{t^{1/4}}\sum_{i,j=1}^\infty e^{-(\lambda_i+\lambda_j)t}\;\left[\int_0^1 (\rho_\epsilon*\eta_i)_\theta\,(\rho_\epsilon*\eta_j)_\theta\, h_\theta\, d\theta\right]^2.$$

Now, since ρ_ϵ is a symmetric convolution kernel:

$$\int_0^1 (\rho_\epsilon*\eta_i)_\theta\,(\rho_\epsilon*\eta_j)_\theta\, h_\theta\, d\theta \;=\; \langle \eta_j, \rho_\epsilon*[h(\rho_\epsilon*\eta_i)]\rangle$$

$$\Longrightarrow \sum_{j=1}^\infty\left[\int_0^1 (\rho_\epsilon*\eta_i)_\theta\,(\rho_\epsilon*\eta_j)_\theta\, h_\theta\, d\theta\right]^2 \;=\; \|\rho_\epsilon*[h(\rho_\epsilon*\eta_i)]\|^2 \;\le\; \|h\|^2,$$

so that:

$$\int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\,\mathbb{E}\left[F(B)\right]\left(c^t_{\epsilon,\theta,\theta'}\right)^2 \;\le\; \frac{\kappa\,\|h\|^2}{t^{1/4}}\sum_{i=1}^\infty e^{-\lambda_i t} \;\le\; \frac{\kappa\,\|h\|^2}{t^{3/4}}.$$

Now for all $\ell \in L$ we have:

$$F(z + s\,\ell) \;=\; F^{-s\,e^{tA}\ell_\theta, -s\,e^{tA}\ell_{\theta'}}(z) \;=\; F^{s\,e^{tA}\ell_\theta, s\,e^{tA}\ell_{\theta'}}(z)$$

so that, setting $H^i := e^{tA} Q k^i$:

$$\begin{aligned} \mathbb{E}\left[\partial^2_{K^1,K^2} F(B)\right] \;&=\; \frac{\partial^2}{\partial r\,\partial s}\mathbb{E}\left[F(B + rK^1 + sK^2)\right]\Big|_{r=s=0} \qquad (6.16) \\ &=\; \frac{\partial^2}{\partial r\,\partial s}\,\Gamma(rH^1_\theta + sH^2_\theta,\, rH^1_{\theta'} + sH^2_{\theta'})\Big|_{r=s=0} \;=\; -\,\frac{\mathbf{v}_1^T\,\Lambda^{-1}_{\theta,\theta'}\,\mathbf{v}_2}{2\pi\sqrt{\det \Lambda_{\theta,\theta'}}} \end{aligned}$$

where $\mathbf{v}_i = (H^i_\theta, H^i_{\theta'}) \in \mathbb{R}^2$. Since the entries of $\Lambda_{\theta,\theta'}$ are bounded uniformly in $\theta, \theta' \in [0,1]$ and for all $\theta, \theta' \in [0,1]$:

$$|H^j_\theta| \;\le\; \sum_{i=1}^\infty \frac{e^{-\lambda_i t}}{\lambda_i}\,\|\rho_\epsilon * e_i'\|_\infty\,\|e_i\|_\infty \;\le\; \sum_{i=1}^\infty \frac{e^{-\lambda_i t}}{\lambda_i^{1/2}} \;\le\; \kappa(1 + |\ln t|), \qquad (6.17)$$

then we obtain:

$$\left|\mathbb{E}\left[\partial^2_{K^1,K^2} F(B)\right]\right| \;\le\; \frac{\kappa(1 + |\ln t|)^2}{(\det \Lambda_{\theta,\theta'})^{3/2}}$$

and therefore:

$$\int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\, \langle K^1, k^2\rangle\, \mathbb{E}\left[\partial^2_{K^1,K^2} F(B)\right] \;\le\; \frac{\kappa(1 + |\ln t|)^2}{t^{3/4}}.$$

Analogously:

$$\begin{aligned} \mathbb{E}\left[\partial^4_{K^1,K^1,K^1,K^2} F(B)\right] \;&=\; \frac{\partial^4}{\partial^2 r\,\partial^2 s}\,\Gamma(rH^1_\theta + sH^2_\theta,\, rH^1_{\theta'} + sH^2_{\theta'})\Big|_{r=s=0} \\ &=\; \frac{1}{(\det \Lambda_{\theta,\theta'})^{3/2}}\, R_{\theta,\theta'}(H^1_\theta, H^2_\theta, H^1_{\theta'}, H^2_{\theta'}), \end{aligned}$$

where $R_{\theta,\theta'}$ is a multi-linear form on $\mathbb{R}^4$ with uniformly bounded coefficients w.r.t. $\theta, \theta' \in [0,1]$. Therefore:

$$\int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\, \mathbb{E}\left[\partial^4_{K^1,K^1,K^1,K^2} F(B)\right] \;\le\; \frac{\kappa(1 + |\ln t|)^4}{t^{3/4}}.$$

Estimate of I_2. Continuing with the notations introduced in the previous step, we notice now that:

$$\frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \cdot \frac{z(t,\theta)}{q_t(\theta)} \cdot \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \cdot \frac{z(t,\theta)}{q_t(\theta)} \;=\; \frac{\partial^2}{\partial a\,\partial b}\, F^{a,b}(z)\Big|_{a=b=0}.$$

Then, setting $\nu_{\epsilon,\theta} := (\rho_\epsilon * q_t(\cdot,\theta))'_\theta$, we have:

$$\begin{aligned} I_2(t,\epsilon) \;&=\; \int \mu(dz) \left[\int_0^1 h_\theta\, \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}}\, 2\, \frac{z(t,\theta)}{q_t(\theta)}\, \nu_{\epsilon,\theta}\, \partial_\theta z_\epsilon(t,\theta)\, d\theta \right]^2 \\ &= 4 \int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\, \nu_{\epsilon,\theta}\, \nu_{\epsilon,\theta'}\; \mathbb{E}\left[\langle B, k^1\rangle\, \langle B, k^2\rangle\; \frac{\partial^2}{\partial a\,\partial b}\, F^{a,b}(B)\Big|_{a=b=0} \right]. \end{aligned}$$

By (6.13) we have:

$$\begin{aligned} &\mathbb{E}\left[\langle B, k^1\rangle\, \langle B, k^2\rangle\, F^{a,b}(B)\right] \\ &= \langle K^1, k^2\rangle\, \mathbb{E}\left[F^{a,b}(B)\right] + \mathbb{E}\left[\partial^2_{K^1,K^2}\, F^{a,b}(B)\right] \\ &= c^t_{\epsilon,\theta,\theta'}\, \Gamma(a,b) \;+\; \frac{\partial^2}{\partial r\,\partial s}\, \Gamma(a - rH^1_\theta - sH^2_\theta,\, b - rH^1_{\theta'} - sH^2_{\theta'})\Big|_{r=s=0}. \end{aligned}$$

Now, recalling that Γ is the density of $\mathcal{N}(0, \Lambda_{\theta,\theta'})$, we can compute:

$$\frac{\partial^2}{\partial a\,\partial b}\, \Gamma(a,b)\Big|_{a=b=0} \;=\; \frac{q^t(\theta,\theta')}{2\pi(\det \Lambda_{\theta,\theta'})^{3/2}},$$

$$\begin{aligned} &\frac{\partial^4}{\partial a\,\partial b\,\partial r\,\partial s} \Gamma(a - rH^1_\theta - sH^2_\theta,\, b - rH^1_{\theta'} - sH^2_{\theta'})\Big|_{a=b=r=s=0} \\ &= \frac{1}{(\det \Lambda_{\theta,\theta'})^{3/2}}\, \widehat{R}_{\theta,\theta'}(1, 1, H^1_\theta, H^2_\theta, H^1_{\theta'}, H^2_{\theta'}) \end{aligned}$$

where $\widehat{R}_{\theta,\theta'}$ is a multi-linear form on $\mathbb{R}^6$ with uniformly bounded coefficients w.r.t. $\theta, \theta' \in [0,1]$. By (6.8):

$$q_t(\theta,\theta') \;=\; \theta \wedge \theta' - \sum_{i=1}^{\infty} \frac{e^{-\lambda_i t}}{\lambda_i}\, e_i(\theta)\, e_i(\theta').$$

Since $(\rho_\epsilon * q_\infty(\cdot,\theta))'_\theta = (\rho_\epsilon * 1_{[0,\theta]})_\theta = 1/2$, then:

$$\nu_{\epsilon,\theta} \;=\; (\rho_\epsilon * q_t(\cdot,\theta))'_\theta = \frac{1}{2} - \sum_{i=1}^{\infty} \frac{e^{-\lambda_i t}}{\lambda_i} (\rho_\epsilon * e'_i)_\theta \; e_i(\theta)$$

and therefore

$$|\nu_{\epsilon,\theta}| \;\leq\; \kappa(1+|\ln t|).$$

Therefore we have proven that:

$$\begin{aligned} I_2(t,\epsilon) \;&\leq\; \kappa \int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\, \nu_{\epsilon,\theta}\, \nu_{\epsilon,\theta'}\, \frac{1+|\widehat{R}_{\theta,\theta'}(1,1,H^1_\theta,H^2_\theta,H^1_{\theta'},H^2_{\theta'})|}{(\det \Lambda_{\theta,\theta'})^{3/2}} \\ &\leq\; \frac{\kappa(1+|\ln t|)^6}{t^{3/4}}. \end{aligned}$$

Estimate of I_3. Arguing like for I_2 we obtain:

$$\begin{aligned} I_3(t,\epsilon) \;&=\; \int \mu(dz) \left[\int_0^1 h_\theta \, \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \left[\frac{(z(t,\theta))^2}{(q_t(\theta))^2} - \frac{1}{q_t(\theta)} \right] \nu^2_{\epsilon,\theta}\, d\theta \right]^2 \\ &=\; \int_{[0,1]^2} d\theta\, d\theta'\, h_\theta\, h_{\theta'}\, \nu^2_{\epsilon,\theta}\, \nu^2_{\epsilon,\theta'} \, \frac{\partial^4}{\partial^2 a\, \partial^2 b} \Gamma(a,b) \Big|_{a=b=0} \;\leq\; \frac{\kappa(1+|\ln t|)^6}{t^{3/4}}, \end{aligned}$$

and the proof of Lemma 5.3 is complete. $\square$

Using the proofs of Proposition 5.1 and Lemma 5.3, we prove also the following:

Corollary 6.1. *For all $h \in C_c(0,1)$, $\mathcal{R}G_\epsilon$ converges weakly in* $\mathrm{Dom}(\mathcal{E})$ *to $\mathcal{R}G_0 \in \mathrm{Dom}(\mathcal{E})$, where for μ-a.e $z \in C$:*

$$\begin{aligned} \mathcal{R}G_0(z) \;:=\; \int_0^\infty \int_0^1 h_\theta \, \frac{e^{-(z(t,\theta))^2/2q_t(\theta)}}{\sqrt{2\pi q_t(\theta)}} \Bigg[\left(\frac{\partial z(t,\theta)}{\partial\theta} \right)^2 - c^t_{0,\theta} \\ -2\,\nu_{0,\theta}\, \frac{z(t,\theta)}{q_t(\theta)}\, \frac{\partial z(t,\theta)}{\partial \theta} + \nu^2_{0,\theta} \left(\left[\frac{z(t,\theta)}{q_t(\theta)} \right]^2 - \frac{1}{q_t(\theta)} \right) \Bigg] d\theta\, dt, \end{aligned}$$

for $\theta \in (0,1)$, $t \in (0,\infty)$, $z(t,\theta)$ is defined by (6.1) and:

$$c^t_{0,\theta} \;:=\; \sum_{i=1}^{\infty} \frac{e^{-\lambda_i t}}{\lambda_i} \, |e'_i(\theta)|^2, \qquad \nu_{0,\theta} \;:=\; \frac{1}{2} - \sum_{i=1}^{\infty} \frac{e^{-\lambda_i t}}{\lambda_i}\, e'_i(\theta)\, e_i(\theta).$$

Moreover for all $F \in \mathrm{Lip}_e(L)$ and $a \in \mathbb{R}$:

$$\mathbb{E}\left[F(B)\int_0^1 h_\theta\ :\dot B_\theta^2:\ dL_\theta^a\right] \tag{6.18}$$

$$= \mathbb{E}[F(B)]\int_0^1 h_\theta\,\frac{a^2-\theta}{4\theta^2}\,\frac{e^{-a^2/2\theta}}{\sqrt{2\pi\theta}}\,d\theta \,+\, \mathcal{E}\Big(e^{a\langle l'',\cdot\rangle}\ F,\ \mathcal{R}G_0\Big)\ e^{-a^2\|\ell'\|^2/2},$$

where $l \in C^2([0,1])$, $l(0)=0$ and $l(x)=1$ for all x such that $h(x)\neq 0$.

Formula (6.18) allows to compute directly the value of the generalized functional constructed in Theorem 2.1 without using the limit in the l.h.s. of (2.2).

7 The case of quadratic φ and constant h

We want to consider the divergence of a vector field of particular interest, namely the identity $\mathcal{K}(\omega)=\omega$. This case corresponds to $\varphi(r)=\frac{1}{2}r^2$ and $h\equiv 1$, and therefore it does not fit in the assumptions of Theorem 2.2, since h has not compact support in $(0,1)$. Still, since $\varphi''\equiv 1$, this case is simpler than the general one and can be treated without the main estimate of Lemma 5.3.

Let us go back to the result of Lemma 3.5: formula (3.7) becomes

$$k_\theta\,\mathbb{E}[B_\theta+K_\theta] \,=\, -\frac{1}{2}\,\frac{d^2}{d\theta^2}\,\mathbb{E}\left[(B_\theta+K_\theta)^2\right] + \mathbb{E}\left[\lambda(\theta,K'_\theta,B_\theta)\right],$$

$$\text{i.e.}\qquad k_\theta\,K_\theta \,=\, -\frac{1}{2}\,\frac{d^2}{d\theta^2}\,(\theta+K_\theta^2) \,+\, (K'_\theta)^2.$$

Integrating over $[0,1]$ in $d\theta$ we obtain:

$$\int_0^1 k_\theta\,K_\theta\,d\theta \,=\, -\frac{1}{2}\left[(K_\theta^2)'\right]_0^1 + \int_0^1 (K'_\theta)^2\,d\theta \,=\, \int_0^1 (K'_\theta)^2\,d\theta,$$

since $K_0=K'_1=0$. By (3.1) this yields for $\Psi_k := e^{\langle k,\cdot\rangle}$:

$$\begin{aligned}\mathbb{E}\left[\partial_B\Psi_k(B)\right] &= e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 (K'_\theta)^2\,d\theta \,=\, \mathbb{E}\left[\Psi_k(B)\int_0^1 :\dot B_\theta^2:\ d\theta\right] \\ &:= \lim_{\epsilon\to 0}\mathbb{E}\left[\Psi_k(B)\int_0^1 :\dot B_{\epsilon,\theta}^2:\ d\theta\right].\end{aligned}$$

In this case $:\dot{B}^2_\theta:$ appears without integration w.r.t. the local time process and is therefore defined in the classical way, see [5]. Arguing like in sections 5 and 6, we set now:

$$\mathcal{G}_\epsilon(B) \ := \int_0^1 :\dot{B}^2_{\epsilon,\theta}: \, d\theta,$$

and we compute for all $z \in C$:

$$P_t\mathcal{G}_\epsilon(z) \ = \int_0^1 \left[(\partial_\theta z_\epsilon(t,\theta))^2 - c_{\epsilon,\theta}\right] d\theta.$$

Arguing like in the proof of Lemma 5.3, see in particular the estimate of I_1, we compute:

$$\|P_t\mathcal{G}_\epsilon\|^2_{L^2(\mu)} \ = \int \mu(dz) \left[\int_0^1 \left[(\partial_\theta z_\epsilon(t,\theta))^2 - c^t_{\epsilon,\theta}\right] d\theta\right]^2 = \int_{[0,1]^2} d\theta\, d\theta' \ \cdot$$

$$\cdot\, \mathbb{E}\left[\langle k^1, B\rangle^2 \langle k^2, B\rangle^2 \ - \ \langle k^1, B\rangle^2 c^t_{\epsilon,\theta'} \ - \ c^t_{\epsilon,\theta} \langle k^2, B\rangle^2 \ + \ c^t_{\epsilon,\theta}\, c^t_{\epsilon,\theta'}\right]$$

$$= \int_{[0,1]^2} d\theta\, d\theta'\, 2\, \langle Qk^1, k^2\rangle^2 \ = 2 \int_{[0,1]^2} d\theta\, d\theta' \left[\sum_{i=1}^\infty \frac{e^{-\lambda_i t}}{\lambda_i} (\rho_\epsilon * e_i')_\theta\, (\rho_\epsilon * e_i')_{\theta'}\right]^2$$

$$= 2 \sum_{i,j=1}^\infty e^{-(\lambda_i+\lambda_j)t} \left[\int_0^1 (\rho_\epsilon * \eta_i)_\theta\, (\rho_\epsilon * \eta_j)_\theta\, d\theta\right]^2 \le \ 2 \sum_{i=1}^\infty e^{-\lambda_i t} \ \le \ \frac{\kappa}{t^{1/2}}.$$

Therefore $\mathcal{R}\mathcal{G}_\epsilon$ converges weakly in $\mathrm{Dom}(\mathcal{E})$ to $\mathcal{R}\mathcal{G}_0 \in \mathrm{Dom}(\mathcal{E})$ and

$$\mathbb{E}[\langle \nabla F(B), B\rangle] \ = \ \lim_{\epsilon \to 0} \mathbb{E}\left[F(B) \int_0^1 :\dot{B}^2_{\epsilon,\theta}: \, d\theta\right] \ = \ \mathcal{E}(F, \mathcal{R}\mathcal{G}_0),$$

for all $F \in \mathrm{Dom}(\mathcal{E})$, where:

$$\mathcal{R}\mathcal{G}_0(z) \ = \int_0^\infty \left[\left(\frac{\partial z(t,\theta)}{\partial \theta}\right)^2 - \ c^t_{0,\theta}\right] dt, \qquad \mu - \text{a.e. } z \in C,$$

see Corollary 6.1 above.

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